Eight-chain model and its variants for hyperelastic rubber-like materials: A comparative study

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ABSTRACT

The eight-chain model, also known as Arruda-Boyce model, is a widely used structural model with micromechanical explanation of its constituent parameters. The energy function of this model is expressed as

$$\Psi = \mu N \left[\gamma \lambda_r + \ln \left[\frac{\gamma}{\sinh \gamma} \right] \right], \tag{1}$$

where μ, N are material parameters and λ_r, γ are relative chain stretch,Langevin function, respectively. Despite its excellent performance to capture the uniaxial and pure shear data of Treloar, it faces a significant deficiency to predict the equibiaxial data. To ameliorate this drawback, over the years, several modified versions of this important model have been proposed. In this contribution, seven such modified versions, e.g. modified Flory-Erman model, Bootstrap 8-chain model, Gornet model, Meissner-Matejka model, Kroon model, Bechir model, Zuniga-Beatty model are compared with the classical eight-chain model. For comparison of all selected models in reproducing the well-known Treloar data , the analytical expressions for the three homogeneous deformation modes, i.e. uniaxial tension (UN), equibiaxial tension (EB) and pure shear (PS) have been derived and the performances of the models are analysed. The analytical expressions for the three deformation modes UN, EB and PS for the eight-chain model, which are required to check the performance of the model on the Treloar data, are derived as

$$P_1^{UN} = \frac{\mu^{UN}}{3} \left[\frac{3N^{UN} - \lambda_{cu}^2}{N^{UN} - \lambda_{cu}^2} \right] \left[\lambda - \lambda^{-2} \right], \quad \lambda_{cu} = \sqrt{\frac{1}{3}[\lambda^2 + \frac{2}{\lambda}]} \tag{2}$$

$$P_{1,2}^{EB} = \frac{\mu^{EB}}{3} \left[\frac{3N^{EB} - \lambda_{cb}^2}{N^{EB} - \lambda_{cb}^2} \right] \left[\lambda - \lambda^{-5} \right], \quad \lambda_{cb} = \sqrt{\frac{1}{3}[2\lambda^2 + \frac{1}{\lambda^4}]} \tag{3}$$

$$P_1^{PS} = \frac{\mu^{PS}}{3} \left[\frac{3N^{PS} - \lambda_{cp}^2}{N^{PS} - \lambda_{cp}^2} \right] \left[\lambda - \lambda^{-3} \right], \quad \lambda_{cp} = \sqrt{\frac{1}{3}[\lambda^2 + \lambda^{-2} + 1]}. \tag{4}$$

By fitting the UN, EB and PS equations to the corresponding Treloar data, the optimal material parameters for the eight-chain model are found:

$$\mu^{UN} = 0.2640 \text{ MPa}, \quad \mu^{EB} = 0.3580 \text{ MPa}, \quad \mu^{PS} = 0.3110 \text{ MPa}$$
$$N^{UN} = 25.600, \quad N^{EB} = 30.260, \quad N^{PS} = 51.320.$$

For comparison, each figure contains the errors between (a) each experiment and its optimal fit, e.g. Error(UN-fit), and (b) the simulations of the other deformation modes and their respective measurements, e.g. Error(EB-sim), cf. Fig (1).

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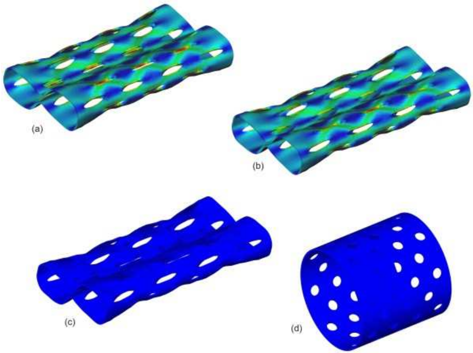
- numerous constitutive models
- which one is the best, in terms of
  - capturing experimental data
  - number of parameters
  - parameter identification
  - micromechanically- or phenomenologically-motivated
- **eight-chain model is one of the most successful models**



MARCKMANN & VERRON, RCT, 2006
BOYCE & ARRUDA, RCT, 2000
REESE ET AL., CMAME, 2010

constitutive models under study

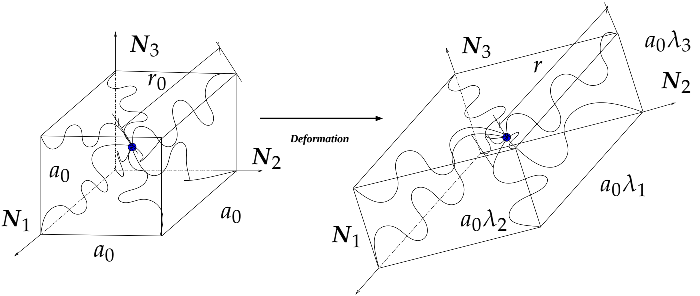
- eight chain model
- modified flory-erman model
- gornet-desmorat (GD) model
- bootstrapped eight-chain model
- meissner-matějka model
- bechir model

salient features

–non-affine micro to macro transformation

–can capture most of the documented data only with two material parameters

–problem mainly in capturing biaxial data



energy function

$$\Psi_{8ch} = \mu N \left[\gamma \lambda_{c,r} + \ln \left(\frac{\gamma}{\sinh \gamma} \right) \right], \lambda_{c,r} = \frac{\lambda_c}{\sqrt{N}} = \sqrt{\frac{I_1}{3N}}$$

uniaxial tension (UN), equibiaxial tension(EQ) and pure shear (PS) formulations

$$P_1^{UN} = \frac{\mu^{UN}}{3}\left[\frac{3N^{UN} - \lambda_{cu}^2}{N^{UN} - \lambda_{cu}^2}\right]\left[\lambda - \lambda^{-2}\right], \quad \lambda_{cu} = \sqrt{\frac{1}{3}\left[\lambda^2 + \frac{2}{\lambda}\right]}$$

$$P_{1,2}^{EB} = \frac{\mu^{EB}}{3}\left[\frac{3N^{EB} - \lambda_{cb}^2}{N^{EB} - \lambda_{cb}^2}\right]\left[\lambda - \lambda^{-5}\right], \quad \lambda_{cb} = \sqrt{\frac{1}{3}\left[2\lambda^2 + \frac{1}{\lambda^4}\right]}$$

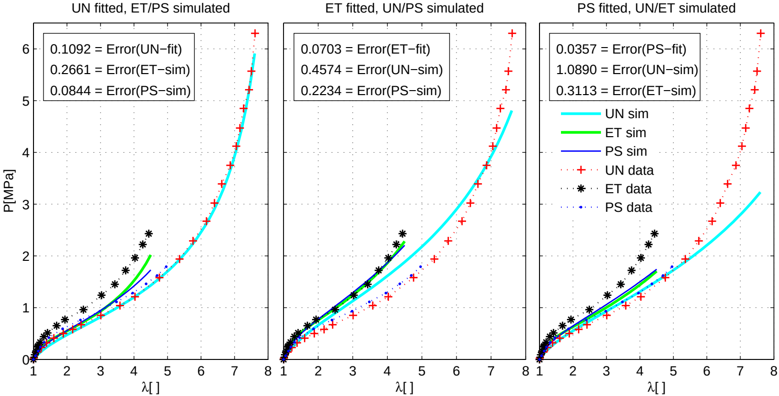
$$P_1^{PS} = \frac{\mu^{PS}}{3}\left[\frac{3N^{PS} - \lambda_{cp}^2}{N^{PS} - \lambda_{cp}^2}\right]\left[\lambda - \lambda^{-3}\right], \quad \lambda_{cp} = \sqrt{\frac{1}{3}\left[\lambda^2 + \lambda^{-2} + 1\right]}.$$

parameter identification

- simultaneous optimization, i.e. inclusion of all experimental data in the parameter identification process
- separate identification for each deformation mode, i.e. uniaxial tension (UN), equibiaxial tension (EB), pure shear (PS)

performance study with Treloar data

$$\text{Error(fit/sim)} = \sqrt{\frac{1}{M}\sum_{i=1}^{M}[P_{\text{fit/sim}}(\lambda_i^{\text{Treloar}}) - P_{\text{Treloar}}(\lambda_i^{\text{Treloar}})]^2},$$



STEINMANN, HOSSAIN, POSSART, Arch. Appl. Mech. 82(9), 2012

salient features

–total energy function is decomposed into two parts, i.e. a cross-link contribution and an entanglement contribution

–eight-chain energy function is taken for the cross-link contribution

total energy function

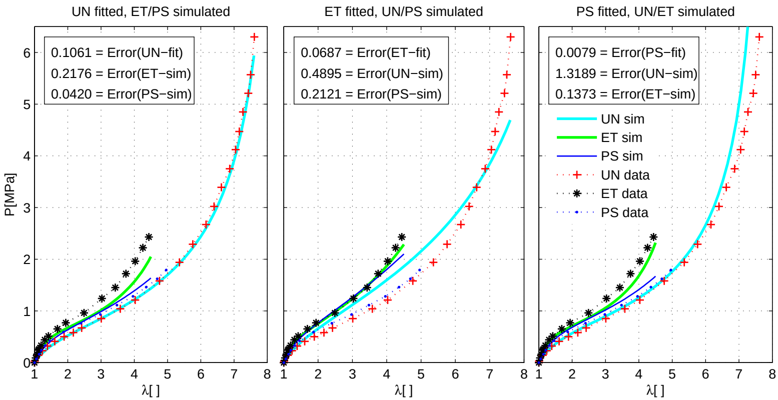
$$\Psi = \Psi_{ph} + \Psi_{ct}.$$

where,

$$\begin{aligned}\Psi_{ct} &= \sum_{i=1}^{3} \frac{\mu}{2}\left[B_i + D_i - \ln(B_i+1) - \ln(D_i+1)\right] \\ B_i &= \kappa^2[\lambda_i^2 - 1][\lambda_i^2 + \kappa]^{-2},\ D_i = \lambda_i^2\kappa^{-1}B_i\end{aligned}$$

$$\begin{aligned}\Psi &= \Psi_{8ch} + \Psi_{ct} \\ &= \mu N\left[\gamma\lambda_{c,r} + \ln\left(\frac{\gamma}{\sinh\gamma}\right)\right] + \sum_{i=1}^{3}\frac{\mu}{2}\left[B_i + D_i - \ln(B_i+1) - \ln(D_i+1)\right]\end{aligned}$$

modified flory-erman model



salient features

–I_1 dependent part relates to Hart-Smith type energy function which is equivalent to eight-chain model

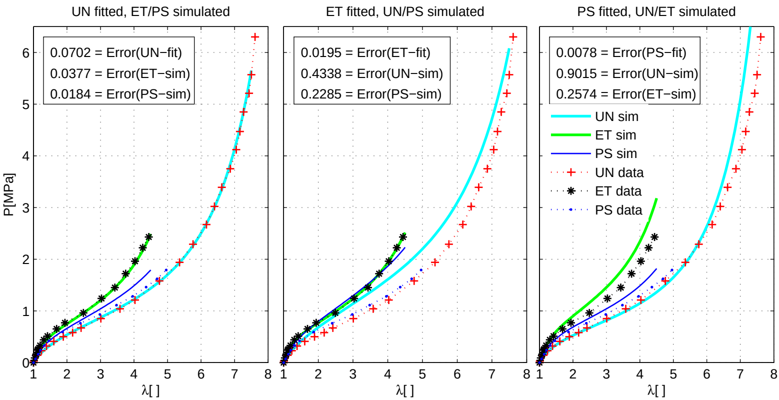
–I_2 dependent part is for entanglement effects

GD energy function

$$\Psi = h_1 \int e^{h_3[I_1-3]^2} dI_1 + 3h_2 \int \frac{1}{\sqrt{I_2}} dI_2$$

UN, EQ and PS formulations

$$\begin{aligned} P_1^{UN} &= 2\left[h_1 e^{h_3[I_1-3]^2} + \frac{3h_2}{\lambda\sqrt{I_2}}\right]\left[\lambda - \lambda^{-2}\right] \\ P_{1,2}^{EB} &= 2\left[h_1 e^{h_3[I_1-3]^2} + \frac{3h_2\lambda^2}{\sqrt{I_2}}\right]\left[\lambda - \lambda^{-5}\right] \\ P_1^{PS} &= 2\left[h_1 e^{h_3[I_1-3]^2} + \frac{3h_2}{\sqrt{I_2}}\right]\left[\lambda - \lambda^{-3}\right]. \end{aligned}$$



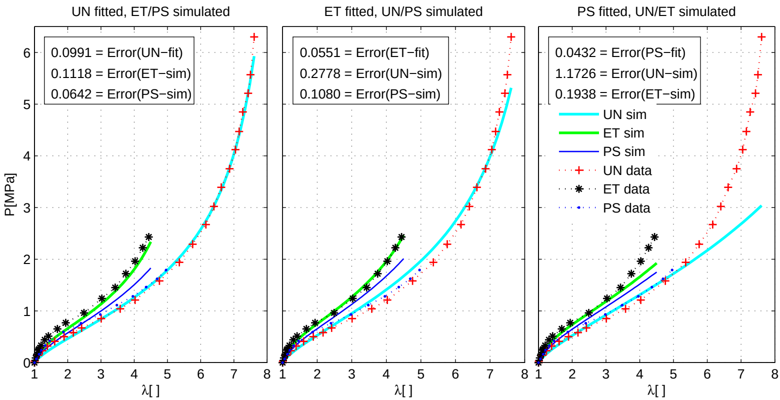
UN fitted, ET/PS simulated
0.0702 = Error(UN−fit)
0.0377 = Error(ET−sim)
0.0184 = Error(PS−sim)
ET fitted, UN/PS simulated
0.0195 = Error(ET−fit)
0.4338 = Error(UN−sim)
0.2285 = Error(PS−sim)
PS fitted, UN/ET simulated
0.0078 = Error(PS−fit)
0.9015 = Error(UN−sim)
0.2574 = Error(ET−sim)
UN sim
ET sim
PS sim
UN data
ET data
PS data
P[MPa]
λ[ ]

energy function

$$\Psi = \Psi_{8ch}\left(\frac{\lambda_c}{\sqrt{N}}\right) + \Psi_{8ch}\left(\frac{i_1}{\sqrt{3N}} - \frac{\lambda_c}{\sqrt{N}}\right)$$

UN, EQ and PS formulations

$$\begin{aligned}
P_1^{UN} &= \frac{\mu^{UN}}{\lambda}\lambda_u \frac{3N^{UN}-\lambda_u^2}{N^{UN}-\lambda_u^2}\left[\frac{\lambda-\lambda^{-1/2}}{\sqrt{3}} - \frac{\lambda^2-\lambda^{-1}}{3\lambda_{cu}}\right] \\
&+ \frac{\mu^{UN}}{3}\left[\frac{3N^{UN}-\lambda_{cu}^2}{N^{UN}-\lambda_{cu}^2}\right]\left[\lambda-\lambda^{-2}\right] \\
P_{1,2}^{EB} &= \frac{\mu^{EB}}{\lambda}\lambda_b \frac{3N^{EB}-\lambda_b^2}{N^{EB}-\lambda_b^2}\left[\frac{\lambda-\lambda^{-2}}{\sqrt{3}} - \frac{\lambda^2-\lambda^{-4}}{3\lambda_{cb}}\right] \\
&+ \frac{\mu^{EB}}{3}\left[\frac{3N^{EB}-\lambda_{cb}^2}{N^{EB}-\lambda_{cb}^2}\right]\left[\lambda-\lambda^{-5}\right] \\
P_1^{PS} &= \frac{\mu^{PS}}{\lambda}\lambda_p \frac{3N^{PS}-\lambda_p^2}{N^{PS}-\lambda_p^2}\left[\frac{\lambda-\lambda^{-1}}{\sqrt{3}} - \frac{\lambda^2-\lambda^{-2}}{3\lambda_{cp}}\right] \\
&+ \frac{\mu^{PS}}{3}\left[\frac{3N^{PS}-\lambda_{cp}^2}{N^{PS}-\lambda_{cp}^2}\right]\left[\lambda-\lambda^{-3}\right].
\end{aligned}$$


UN fitted, ET/PS simulated
0.0991 = Error(UN−fit)
0.1118 = Error(ET−sim)
0.0642 = Error(PS−sim)
ET fitted, UN/PS simulated
0.0551 = Error(ET−fit)
0.2778 = Error(UN−sim)
0.1080 = Error(PS−sim)
PS fitted, UN/ET simulated
0.0432 = Error(PS−fit)
1.1726 = Error(UN−sim)
0.1938 = Error(ET−sim)
UN sim
ET sim
PS sim
UN data
ET data
PS data
P[MPa]
λ[ ]

meissner-matějka model

salient features

–entanglement part is the same as in extended tube model by kaliske et al [1999]

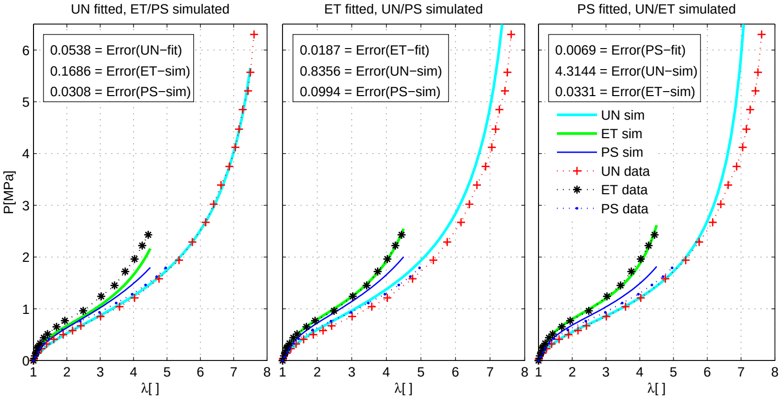
–cross-link part is taken from eight-chain energy function

energy function

$$\Psi = \mu N \left[\gamma \lambda_{c,r} + \ln \left(\frac{\gamma}{\sinh \gamma} \right) \right] + \sum_{i=1}^{3} \frac{2\mu_e}{\beta^2} \left[\lambda_i^{-\beta} - 1 \right]$$

UN, EQ and PS formulations

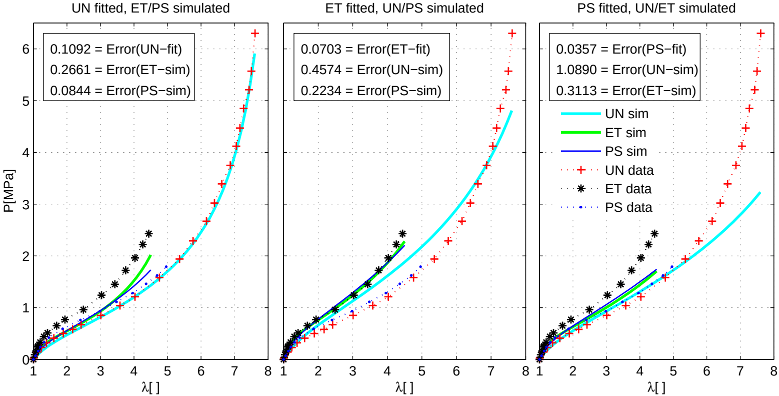
$$\begin{aligned} P_1^{UN} &= \frac{\mu^{UN}}{3} \left[\frac{3N^{UN} - \lambda_{cu}^2}{N^{UN} - \lambda_{cu}^2} \right] \left[\lambda - \lambda^{-2} \right] + \frac{2\mu_e^{UN}}{\beta^{UN}} \left[\lambda^{\frac{\beta^{UN}}{2} - 1} - \lambda^{-\beta^{UN} - 1} \right] \\ P_{1,2}^{EB} &= \frac{\mu^{EB}}{3} \left[\frac{3N^{EB} - \lambda_{cb}^2}{N^{EB} - \lambda_{cb}^2} \right] \left[\lambda - \lambda^{-5} \right] + \frac{2\mu_e^{EB}}{\beta^{EB}} \left[\lambda^{2\beta^{EB} - 1} - \lambda^{-\beta^{EB} - 1} \right] \\ P_1^{PS} &= \frac{\mu^{PS}}{3} \left[\frac{3N^{PS} - \lambda_{cp}^2}{N^{PS} - \lambda_{cp}^2} \right] \left[\lambda - \lambda^{-3} \right] + \frac{2\mu_e^{PS}}{\beta^{PS}} \left[\lambda^{\beta^{PS} - 1} - \lambda^{-\beta^{PS} - 1} \right] \end{aligned}$$


UN fitted, ET/PS simulated
0.0538 = Error(UN−fit)
0.1686 = Error(ET−sim)
0.0308 = Error(PS−sim)
ET fitted, UN/PS simulated
0.0187 = Error(ET−fit)
0.8356 = Error(UN−sim)
0.0994 = Error(PS−sim)
PS fitted, UN/ET simulated
0.0069 = Error(PS−fit)
4.3144 = Error(UN−sim)
0.0331 = Error(ET−sim)
UN sim
ET sim
PS sim
UN data
ET data
PS data
P[MPa]
λ[ ]

$$\begin{aligned}\Psi &= \Psi_{8ch} + \Psi_{3ch} \\ &= \mu_f N_8 \left[\lambda_{c,r}\beta + \ln\left(\frac{\beta}{\sinh\beta}\right)\right] + \frac{\mu_c N_3}{3}\sum_{j=1}^{j=3}\left[\bar{\lambda}_{rj}\bar{\beta}_j + \ln\left(\frac{\bar{\beta}_j}{\sinh\bar{\beta}_j}\right)\right]\end{aligned}$$

UN, EQ and PS formulations

$$\begin{aligned}P_1^{UN} &= \left[1 - \frac{\eta^{UN}\lambda_k}{\sqrt{N_3^{UN}}}\right]\frac{\mu^{UN}}{3\lambda}\left[\lambda_k^2\frac{3N_3^{UN}-\lambda_k^2}{N_3^{UN}-\lambda_k^2} - \lambda_k^{-1}\frac{3N_3^{UN}-\lambda_k^{-1}}{N_3^{UN}-\lambda_k^{-1}}\right] \\ &+ \frac{\rho^{UN}\mu^{UN}\lambda_{c,ru}}{3}\left[\frac{3-\lambda_{ru}^2}{1-\lambda_{ru}^2}\right]\left[\lambda-\lambda^{-2}\right] \\ P_{1,2}^{EB} &= \left[1 - \frac{\eta^{EB}\lambda_k}{\sqrt{N_3^{EB}}}\right]\frac{\mu^{EB}}{3\lambda}\left[\lambda_k^2\frac{3N_3^{EB}-\lambda_k^2}{N_3^{EB}-\lambda_k^2} - \lambda_k^{-4}\frac{3N_3^{EB}-\lambda_k^{-4}}{N_3^{EB}-\lambda_k^{-4}}\right] \\ &+ \frac{\rho^{EB}\mu^{EB}\lambda_{c,rb}}{3}\left[\frac{3-\lambda_{rb}^2}{1-\lambda_{rb}^2}\right]\left[\lambda-\lambda^{-5}\right] \\ P_1^{PS} &= \left[1 - \frac{\eta^{PS}\lambda_k}{\sqrt{N_3^{PS}}}\right]\frac{\mu^{PS}}{3\lambda}\left[\lambda_k^2\frac{3N_3^{PS}-\lambda_k^2}{N_3^{PS}-\lambda_k^2} - \lambda_k^{-2}\frac{3N_3^{PS}-\lambda_k^{-2}}{N_3^{PS}-\lambda_k^{-2}}\right]\end{aligned}$$



HOSSAIN AND STEINMANN, J. Mech. Behav. Mat., In press, 2012

summary

- a systematic comparative study on the eight-chain model and its modified versions is presented
- most of the modified versions add extra energy function to incorporate the influence of entanglement effects of polymer chains
- among five modified versions meissner-matejka and bootstrapped eight-chain models yield better results in all three deformation modes in the case of Treloar experimental data

references

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Archive of Applied Mechanics, 82(9), 1187-1217, 2012.

thank you for your attention :)

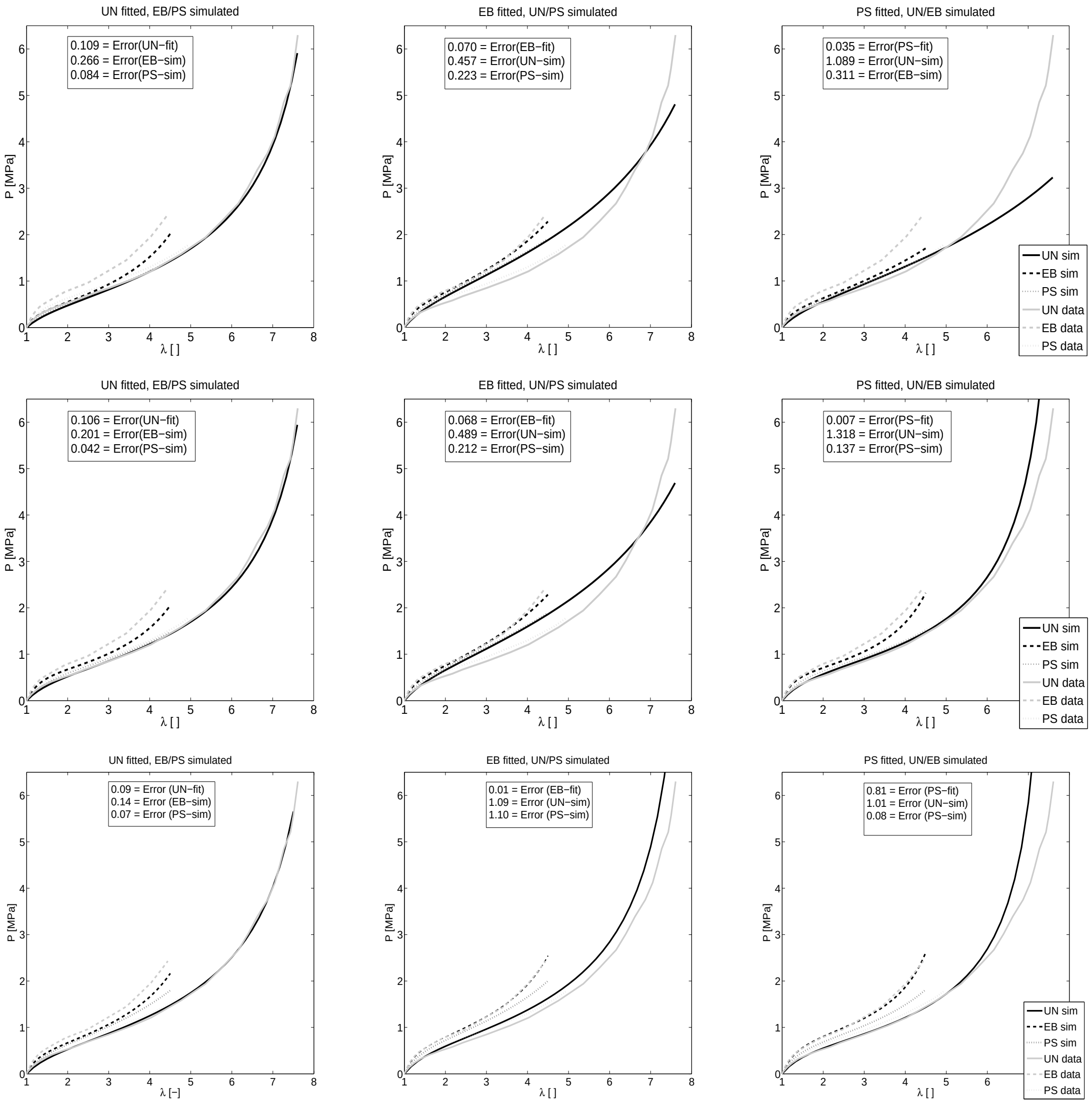

Figure 1: Performance of two modified versions of the 8-chain on the Treloar data. For the validation of each model, each set of optimal material parameters for UN, EB and PS is used to compute the response of the other two deformation modes. (**Top**) Classical 8-chain model, (**Middle**) modified Flory-Erman model, and (**Bottom**) Meissner-Matejka model.

References

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